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Skew-Symmetric Functions on the Hyperboloid and Quantum Measures

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Measures on the logic of J-projections on an indefinite metric space of dimension two are studied.

1. INTRODUCTION

A *quantum logic* (= *orthomodular poset*) is a set E with a partial order $\leq$ and a unary operation $^\perp$ such that (i) E possesses a least and a greatest element, 0 and 1, $0 \neq 1$; (ii) $a \leq b$ implies $b^\perp \leq a^\perp$, $\forall a, b \in E$; (iii) $(a^\perp)^\perp = a$, $\forall a \in E$; (iv) if $a \leq b$, then $b = a \vee (b \wedge a^\perp)$.

In Matvejchuk (1995b), a universal method for constructing projection quantum logics was given. Let $\mathcal{P}$ be a quantum logic of projections on a Hilbert space H with the order $p \leq q$ iff $pq = qp = p$ and orthocomplementation $p^\perp \equiv I - p$. Note that $p = q + e$, $p, q, e \in \mathcal{P}$, implies $eq = qe = 0$. A *quantum measure* (= finite additive measure) is a function μ: $\mathcal{P} \to C$ such that $\mu(e + q) = \mu(e) + \mu(q)$ whenever $eq = qe = 0$. If $\mu \geq 0$ and $\mu(I) = 1$, then μ is said to be a *probability measure* (= *quantum probability measure*).

Problem: Give a description of quantum measures on a quantum logic of projections, is there an extension of a quantum measure to a linear functional on the algebra of bounded operators generated by $\mathcal{P}$?

An important interpretation of a quantum logic is the set Π of all orthogonal projections in a von Neumann algebra $\mathcal{M}$ (or, more generally, in a JW-algebra or an AW^*-algebra). The Mackey–Gleason problem asked:

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when can a countably additive probability measure on Π in a separable Hilbert space be extended to a bounded linear functional on $\mathcal{M}$?

We have the following theorem:

Let $\mathcal{M}$ be a JW-algebra (an AW^*-algebra which has a faithful normal center-valued trace) which has no direct summand of the type I_2. Let μ: $\Pi \to C$ be a bounded quantum measure on the set of all orthogonal projections in $\mathcal{M}$. Then μ has a unique extension to a bounded linear functional on $\mathcal{M}$.

A sketch of the proof was given in Matvejchuk (1988). A complete solution was obtained in Matvejchuk (1987, 1995).

There is an unhappy history of incomplete proofs and fallacious arguments associated with attempts to generalize Gleason's theorem. The above theorem was repeated in a particular case of von Neumann algebras by Bunce and Wright (1992a,b).

The first major step was the work of Gleason (1957). His profound work, which was fundamental for all subsequent advances in this area, considered positive, countably additive quantum measure on $B(H)$, where H is a separable Hilbert space and dim $H \geq 3$. The solution for a von Neumann algebra of type III or II_∞ and for a positive quantum measure was first given by the conjunction of the work of Christensen (1982) and the one for countably additive positive measures for semifinite von Neumann algebras (Matvejchuk, 1980). Later, this result was repeated with a similar proof (Yeadon, 1993).

The problem of the construction of a quantum field theory leads to the indefinite metric spaces (Dadashan and Horujy, 1983). Indefinite metric spaces yield a wide class of projection quantum logics (Matvejchuk, 1995b). In the indefinite case, the set $\mathcal{P}$ of all J-orthogonal projections serves as an analog to the logic Π. There is an indefinite analog to the Gleason theorem (Matvejchuk, 1991a,b; also see Matvejchuk, n.d.):

Let H be a J-space, dim $H \geq 3$, and let μ: $\mathcal{P} \to \mathcal{R}$ be an indefinite measure. Then there exist a J-self-adjoint trace class operator T and a semitrace μ_0 such that $\mu(p) = \mathrm{Tr}(Tp) + \mu_0(p)$, $\forall p \in \mathcal{P}$. Moreover, if the indefinite rank of H is equal to $+\infty$, then $\mu_0(\cdot) \equiv 0$.

2. SOME NOTATION

Let H be a space with an indefinite metric $[.,.]$, a canonical decomposition $H = H^+[\dotplus]H^-$, and a canonical symmetry J. Following the terminology of Azizov and Iokhvidov (1989), H is a *Krein space* (sometimes H is called a *J-space*). H is a Hilbert space with respect to the inner product $(x, y) = [Jx, y]$. Note that $(x, y) = [x_+, y_+] - [x_-, y_-]$, where $x_+, y_+ \in H^+$, $x_-, y_- \in H^-$, and $x = x_+ + x_-$, $y = y_+ + y_-$. There exist orthogonal projections Q^+ and

Q^- such that $I = Q^+ + Q^-$, $J = Q^+ - Q^-$, and $Q^+H = H^+$, $Q^-H = H^-$, $[x, y] = (Jx, y)$, $\forall x, y \in H$. Conversely, let H be a Hilbert space with the inner product $(., .)_1$ and let P be an orthogonal projection with $0 < P < I$. Then H, with respect to $[x, y]_1 \equiv ((2P - I)x, y)_1$, is a J_1-space ($J_1 \equiv 2Q - I$) with the indefinite metric $[., .]_1$.

Let $b \in B(H)$. It is easy to see that p is J-self-adjoint (i.e., $[bx, y] = [x, by]$, $\forall x, y \in H$) $\Leftrightarrow b = Jb^*J$. Note that b is J-self-adjoint $\Leftrightarrow bJ$ is self-adjoint in the Hilbert space H. Every $b \in B(H)$ is the sum, $b = \frac{1}{2}(b + Jb^*J) + (1/2i)(b - Jb^*J)$ of J-self-adjoint operators. Let $\mathcal{P} = \{p \in B(H): p^2 = p \text{ and } [px, y] = [x, py], \forall x, y \in H\}$. The set $\mathcal{P}$ is a quantum logic. A vector $z \in H$ is said to be *positive* (*negative*) if $[z, z] > 0$ ($[z, z] < 0$). The set $\Gamma = \Gamma^+ \cap \Gamma^-$, where $\Gamma^+ \equiv \{f \in H: [f, f] = 1\}$ and $\Gamma^- \equiv \{f \in H: [f, f] = -1\}$ is an analog to the unit sphere $S = \{f \in H: (f, f) = 1\}$. Every one-dimensional projection in $\mathcal{P}$ can be represented in the form $p_f = [f, f][., f]f$, $f \in \Gamma$, and $\|p_f\| = \|f\|^2$. Hence $p_fJ/\|p_fJ\|$ is the orthogonal projection onto subspace $\{\lambda f\}_{\lambda \in C}$. Note that $f \in \Gamma^+ \Leftrightarrow Jp_f \geq 0$, and $f \in \Gamma^- \Leftrightarrow Jp_f \leq 0$. Denote by $\mathcal{P}_1$ the set of all one-dimensional projections in $\mathcal{P}$.

Suppose that $H = R^3$ with the Euclidean inner product. Let P be the orthogonal projection onto the axis OX, and $J_1 = 2P - I$. Then Γ^+ is the two-sheeted hyperboloid, $\{(x, y, z) \in R^3: x^2 - (y^2 + z^2) = 1\}$, and $\Gamma^- = \{(x, y, z) \in R^3: y^2 + z^2 - x^2 = 1\}$ is the hyperboloid of one sheet. Therefore, in the indefinite case $\mathcal{P}$ could be called a hyperbolic logic.

3. THE MAIN RESULTS

It follows from the above that type I_2 is the only obstruction in the problem of the description of a quantum measure, having a positive answer for all other cases. Why does it fail for $M_2(C)$, the algebra of two-by-two complex matrices?

Let H be a two-dimensional complex Krein space. Let $e_+ \in H^+$ and $e_- \in H^-$ be such that $(e_+, e_+) = ([e_+, e_+]) = 1$ and $(e_-, e_-) = 1$. By fixing the orthonormal base e_+, e_- in the underlying Hilbert space, we may identify the algebra $\mathcal{M}$ of all linear operators on H with $M_2(C)$. When

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

we define $\tau W = \frac{1}{2}(w_{11} + w_{22})$. We have $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in the base e_+, e_-. Hence

$$\text{an operator } T \text{ is } J\text{-self-adjoint} \Leftrightarrow T = \begin{pmatrix} a & b + ic \\ -b + ic & d \end{pmatrix}$$

where $a, b, c, d \in R$. Let $\mathcal{M}_h$ be the set of all J-self-adjoint operators, and let $\mathcal{M}_0 \equiv \{T \in \mathcal{M}_h: \tau(T) = 0\}$. We have $T = T_0 + \tau(T)I$, where $T_0 \in \mathcal{M}_0$, $\forall T \in \mathcal{M}_h$. Let $S_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $S_2 \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then $T = \tau(T)I + aJ + bS_1 + cS_2$, $\forall T \in \mathcal{M}_h$. Let ψ be the map $(a, b, c) \to aJ + bS_1 + cS_2$ from R^3 onto $\mathcal{M}_0$. It is evident that ψ is a continuous linear function on R^3 and $\psi(a, b, c)^* = \psi(a, -b, -c)$. It is easy to verify that $\|\psi(a, b, c)\| = |a| + |b + ic|$. Hence the following proposition holds.

Proposition 1. The function ψ realizes a bijection of the gyroscope $\{(a, b, c): |a| + |b + ic| = 1\}$ onto the unit sphere of $\mathcal{M}_0$.

3.1. Properties of the Map ψ on $\Gamma^+ = \{(a, b, c): a^2 - b^2 - c^2 = 1\}$

Lemma 2. $aJ + bS_1 + cS_2 = P - P^\perp$ $(= 2P - I)$, where $P \in \mathcal{P}_1 \Leftrightarrow a^2 - b^2 - c^2 = 1$.

Proof. Let $aJ + bS_1 + cS_2 = 2P - I$, where $P \in \mathcal{P}_1$. Then

$$P = \frac{1}{2}\begin{pmatrix} a + 1 & b + ic \\ -b + ic & 1 - a \end{pmatrix}$$

As $P^2 = P$, we have $a^2 - b^2 - c^2 = 1$.

Conversely, let $a^2 - b^2 - c^2 = 1$. Put

$$P \equiv \frac{1}{2}\begin{pmatrix} a + 1 & b + ic \\ -b + ic & 1 - a \end{pmatrix}$$

It is easy to see that $P^2 = P$ and $P = JP^*J$. Hence $P \in \mathcal{P}_1$ and $2P - I = aJ + bS_1 + cS_2$. ■

Remark 3. $\|P\| = \|JP\| = |a|$ when $\psi(a, b, c) = 2P - I$, $P \in \mathcal{P}_1$.

Remark 4. Let $\psi(a, b, c) = 2P - I$, where $P \in \mathcal{P}_1$. Then $JP \geq 0 \Leftrightarrow a \geq 1$, and $JP \leq 0 \equiv a \leq -1$.

Remark 5. Let $a^2 - b^2 - c^2 = 1$. Then $(\frac{1}{2}(\psi(a, b, c) - I))^\perp = \frac{1}{2}(\psi(-a, -b, -c) - I)$.

It follows from Lemma 3 that ψ maps the hyperboloid Γ^+ of R^3 onto $\{2P - I: P \in \mathcal{P}_1\}$. By Remark 5, when $x \in \Gamma^+$ and $\psi(x) = 2P - I$, then $\psi(-x) = 2P^\perp - I$.

Let V be the set of all real-valued functions ϕ on Γ^+ in R^3, such that $\phi(-x) = -\phi(x)$, $\forall x \in \Gamma^+$. For each $\phi \in V$ we define μ_ϕ on $\mathcal{P} \subset M_2(C)$ by $2\mu_\phi(P) \equiv \phi(\psi^{-1}(2P - I)) + 1$ whenever $P \in \mathcal{P}_1$ and $\mu_\phi(0) = 0$, $\mu_\phi(I) = 1$. Note that

$$\mu_\phi(P) - \mu_\phi(P^\perp) = \tfrac{1}{2}[\phi(\psi^{-1}(P - P^\perp)) + 1 - \phi(\psi^{-1}(P^\perp - P)) - 1]$$
$$= \tfrac{1}{2}[\phi(\psi^{-1}(P - P^\perp)) - \phi(\psi^{-1}(P^\perp - P))] = \phi(\psi^{-1}(P - P^\perp))$$

We have

$$2\mu_\phi(P) + 2\mu_\phi(P^\perp) = \phi(\psi^{-1}(2P - I)) + \phi(-\psi^{-1}(2P - I)) + 2$$
$$= 2, \qquad \forall P \in \mathcal{P}_1$$

Thus $\mu_\phi(P) + \mu_\phi(P^\perp) = \mu_\phi(I)$. Hence μ_ϕ is a quantum measure. Also, $2\mu_\phi(P) \geq -1 + 1 = 0$ if $|\phi(x)| \leq 1$. In this case, μ_ϕ is a quantum probability measure on the J-orthogonal projections $\mathcal{P}$ in M.

Conversely, given a quantum measure μ on $\mathcal{P}$, we may define ϕ on Γ^+ as follows. For each $x \in \Gamma^+$ there is $P \in \mathcal{P}_1$ such that $\psi(x) = 2P - I$. Let $\phi(x) \equiv \mu(P) - \mu(P^\perp)$. Then we see that $\phi(-x) = -\phi(x)$. In addition, $|\phi(x)| \leq \mu(P) + \mu(P^\perp) = \mu(I) = 1$ if μ is a positive probability measure. It is easy to verify that $\mu_\phi = \mu$. We have thus established the following:

Theorem 6. For each $\phi \in V$ and $|\phi| \leq 1$, μ_ϕ is a quantum probability measure on $\mathcal{P} \subset M$. Conversely, every quantum probability measure on $\mathcal{P} \subset M$ arises in this way.

Put $\mathcal{P}^+ \equiv \{P \in \mathcal{P}_1: JP \geq 0\}$ and $\mathcal{P}^- \equiv \{\mathcal{P} \in (\mathcal{P}_1: JP \leq 0\}$.

Example 7. Let $\phi \in V$ be such that $\phi(a, b, c) \equiv 1$ if $a \geq 1$. Then $\mu_\phi/\mathcal{P}^+ \equiv 1$ and $\mu_\phi/\mathcal{P}^- \equiv 0$.

In the terminology of (Matvejchuk, 1991a,b, n.d.), each measure with this property is said to be a *semitrace* (= *semiconstant*) *measure.*

Theorem 8. Let $\phi \in V$. Then μ_ϕ is continuous if and only if ϕ is continuous.

Proof. We may identify Γ^+ with $\{2P - I: P \in \mathcal{P}_1\}$. Since 0 and I are isolated points in $\mathcal{P}$, μ_ϕ is continuous at 0 and I. Let $P \in \mathcal{P}_1$. Let $\|P_n - P\| \to 0$. Then $\|(2P_n - I) - (2P - I)\| \to 0$. If ϕ is continuous at $2P - I$, then $\phi(2P_n - I) \to \phi(2P - I)$. So

$$\mu_\phi(P_n) = \tfrac{1}{2}[\phi(2P_n - I) + 1] \to \tfrac{1}{2}[\phi(2P - I) + 1] = \mu_\phi(P)$$

So μ_ϕ is continuous at P.

Conversely, suppose that μ_ϕ is continuous at P. Then $\phi(2P_n - I) + 1 \to \phi(2P - I) + 1$. So ϕ is continuous at $2P - I$. ■

3.2. Properties of the Operators $\psi(a, b, c)$, $(a, b, c) \in K \equiv \{(a, b, c): a^2 = b^2 + c^2\}$

It is easy to verify that the operator

$$P_\theta \equiv \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix}$$

is an orthogonal projection ($\neq 0$), $JP_\theta J = P_{\theta+\pi}$ [i.e., $(JP_\theta)^* = JP_{\theta+\pi}$] and P_θ, $P_{\theta+\pi}$ are mutually orthogonal. Hence $(JP_\theta)^2 = JP_\theta JP_\theta = 0$ and $JP_\theta \in \mathcal{M}_0$.

Conversely, let

$$T \equiv \begin{pmatrix} a & b + ic \\ -b + ic & -a \end{pmatrix}, \quad (\in \mathcal{M}_0), \qquad T \neq 0, \qquad T^2 = 0$$

Then JT is a self-adjoint operator, $a^2 = b^2 + c^2$, and $\|JT\| = 2|a|$. Hence $Q \equiv JT/(2a)$ is an orthogonal projection, and JQJ and Q are mutually orthogonal projections. Thus we have proved the following

Lemma 9. The map ψ realizes a bijection of the cone K onto $\{T \in \mathcal{M}_0: T^2 = 0\}$. If $(a, b, c) \in K$ and $(a, b, c) \neq (0, 0, 0)$, then $\|\psi(a, b, c)\| = 2|a|$, $Q \equiv [1/(2a)]J\psi(a, b, c)$ is an orthogonal projection, and $JQJQ = 0$.

We see that

$$\begin{aligned} A &\equiv |b + ic|\begin{pmatrix} a & e^{i\theta} \\ -e^{-i\theta} & -a \end{pmatrix} \\ &= |b + ic|\left(\frac{a + 1}{2}\begin{pmatrix} 1 & e^{i\theta} \\ -e^{-i\theta} & -1 \end{pmatrix} + \frac{a - 1}{2}\begin{pmatrix} 1 & -e^{i\theta} \\ e^{-i\theta} & -1 \end{pmatrix}\right) \end{aligned}$$

Let P_θ be the orthogonal projection. Then $A = |b + ic|((a + 1)JP_\theta + (a - 1)JP_{\theta+\pi})$. The operator JA is self-adjoint and $JA = |b + ic|((a + 1)P_\theta + (a - 1)P_{\theta+\pi})$ is the spectral decomposition for JA. Hence by the uniqueness of the spectral decomposition, we have the following.

Lemma 10. For every $A \in \mathcal{M}_0$ there exist a unique operator JP_θ ($\in \{T \in \mathcal{M}_0: T^2 = 0\}$) and numbers $t, d \in R$ such that $A = tJP_\theta + dJP_\theta^\perp$.

3.3. The Linearity of a Quantum Measure

For each $\phi \in V$ we define $\overline{\phi}$ on $N \equiv \{(a, b, c): d^2 \equiv a^2 - b^2 - c^2 > 0, d > 0\} \cup \{(0, 0, 0)\}$ by $\phi(0, 0, 0) = 0$ and, for $(a, b, c) \neq (0, 0, 0)$, $\overline{\phi}(a, b, c) \equiv d\phi(d^{-1}(a, b, c))$.

Now, consider ϕ such that there exists $\lim \overline{\phi}(x_n) \equiv \overline{\phi}(y)$, $\forall y \in K$, and $\forall \{x_n\} \subset N$, $x_n \to y$. Let (a, b, c) be such that $a^2 - b^2 - c^2 < 0$. Put $\overline{\phi}(a,$

$b, c) \equiv \bar{\phi}(a_1, b_1, c_1) + \bar{\phi}(a_2, b_2, c_2)$, where $(a_i, b_i, c_i) \in K$, $i = 1, 2$, and $(a, b, c) = (a, b, c) + (a, b, c)$. By definition, $\bar{\phi}(ta, tb, tc) = t\bar{\phi}(a, b, c)$.

Let $T \in \mathcal{M}_h$ and $T = \tau(T)I + T_0$. Then $T_0 \in \mathcal{M}_0$ and there is a unique triple (a, b, c) such that $\psi(a, b, c) = T_0$. Put $\bar{\mu}_\phi(T) \equiv \tau(T) + \bar{\phi}(\psi^{-1}(T_0))$. Let $T = aP + bP^\perp$, $P \in \mathcal{P}_1$. Then

$$T = \left[\frac{a+b}{2} I + \frac{a-b}{2}(2P - I)\right]$$
$$= \frac{1}{2}[a(P - P^\perp) + a + b(P^\perp - P) + b]$$

Hence

$$\bar{\mu}_\phi(T) = \frac{a+b}{2} + \bar{\phi}\left(\psi^{-1}\left(\frac{a-b}{2}(2P - I)\right)\right)$$
$$= \frac{a+b}{2} + \frac{a-b}{2}\phi(\psi^{-1}(2P - I))$$
$$= \frac{a}{2}[\phi(\psi^{-1}(2P - I)) + 1] + \frac{b}{2}[1 - \phi(\psi^{-1}(2P - I))]$$
$$= \frac{a}{2}[\phi(\psi^{-1}(2P - I) + 1)] + \frac{b}{2}[\phi(\psi^{-1}(2P^\perp - I) + 1)]$$
$$= a\mu_\phi(P) + b\mu_\phi(P^\perp)$$

Thus $\bar{\mu}_\phi$ is an extension of μ_ϕ over $\mathcal{M}_h$. Put

$$\bar{\mu}_\phi(T) \equiv \bar{\mu}_\phi(\tfrac{1}{2}(T + JT^*J) + \bar{\mu}_\phi(\tfrac{1}{2i}(T - JT^*J)), \qquad \forall T \in \mathcal{M}$$

So $\bar{\mu}_\phi$ is linear (continuous) on $\mathcal{M}$ if and only if $\bar{\phi}$ is linear (continuous) on R^3. We have thus established the following:

Theorem 11. The quantum measure μ_ϕ has a linear extension to $\mathcal{M}$ if and only if ϕ has a linear extension to R^3. The functional $\bar{\mu}_\phi$ is continuous on $\mathcal{M}$ if and only if $\bar{\phi}$ is continuous on R^3.

A measure μ is said to be *linear* if there is a linear functional f_μ on $\mathcal{M}$ such that $\mu = f_\mu$ on $\mathcal{P}$.

Theorem 12. Let $\phi \in V$ be a bounded function. Then μ_ϕ is a linear quantum measure if and only if $\phi = 0$. If $\phi = 0$, then $\mu_\phi = \tau$ on $\mathcal{P}$.

Proof. Let $\phi \in V$ be a bounded function. Then by definition, $\bar{\phi} = 0$ on K (and hence $\bar{\phi} = 0$ on $R^3 \backslash N$). Let μ_ϕ be a linear measure. By Theorem 11, ϕ has a linear extension. For every $(a, b, c) \in \Gamma^+$ there is (a_1, b_1, c_1),

$(a_2, b_2, c_2) \in K$ such that $(a, b, c) = (a_1, b_1, c_1) + (a_2, b_2, c_2)$. Hence $\phi(a, b, c) = \overline{\phi}(a_1, b_1, c_1) + \overline{\phi}(a_2, b_2, c_2) = 0$.

Conversely, let $\phi = 0$. By definition, $\mu_\phi(P) = 1/2$, $\forall P \in \mathcal{P}_1$. Hence $\mu_\phi = \tau$ on $\mathcal{P}$. ∎

Let $T \in \mathcal{M}_h$, $T \neq I$, and $\tau(T) = 1$. Then the function ϕ, where $\phi(\psi^{-1}(P - P^\perp)) \equiv \tau(T(P - P^\perp))$, $\forall P \in \mathcal{P}_1$, is unbounded on Γ^+.

3.4. The Relationship Between Measures on the Logics $\mathcal{P}$ and Π

It easy to verify that $\{PJ/\|PJ\|: P \in \mathcal{P}_1\} = \Pi \setminus \{P_\theta\}$. Let μ be a linear measure on $\mathcal{P}$, and let f_μ be the linear function such that $\mu(P) = f_\mu(P)$, $\forall P \in \mathcal{P}$. Put

$$\nu_\mu\left(\frac{PJ}{\|PJ\|}\right) \equiv \frac{1}{\|PJ\|}\,\mu(P) = f_\mu\left(\frac{1}{\|PJ\|}\,(PJ)J\right), \qquad \forall P \in \mathcal{P}_1$$

and $\nu_\mu(0) = 0$, $\nu_\mu(I) = 1$. Then it is clear that ν_μ has a unique extension over Π, and this extension is a linear measure on Π.

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